Note on the gravitational anomaly*

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Abstract

I briefly discuss the gravitational anomaly in two dimensions.

*Based on a talk given at http://www.math-phys.ir/.

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Contents

1 Introduction

In the context of AdS_3/CFT_2, gravitational anomalies are encountered in Topologically Massive Gravity [1]. This anomaly implies that in the boundary CFT $c_L \neq c_R$ [2]. In [3], it is shown that in such models, the boundary stress-energy tensor can be reproduced by integrating the conformal and gravitational anomalies.

Historically, Alvarez-Gaumé and Witten have shown that one-loop graphs describing the propagation of chiral fermions in an external gravitational field in two-dimensions contain gravitational anomalies [4]. Coupling the two-dimensional conformal matter with $c_L \neq c_R$ to two-dimensional gravity is studied in [5].

In this note, following [4] and [6], we study the gravitational anomalies that appear when two-dimensional conformal matter with $c_L \neq c_R$ is coupled to two-dimensional gravity. The interested reader may consult [2] and [3] for further study.

2 Flat space

A key ingredient of CFT is the energy momentum tensor, which, classically is defined by,[1]

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{2.1}$$

For a CFT on a cylinder,

$$\langle T \rangle = 0, \qquad \langle \bar{T} \rangle = 0 \tag{2.2}$$

in which,

$$T = T_{zz}, \qquad \bar{T} = T_{\bar{z}\bar{z}}, \tag{2.3}$$

[1] Note that our convention differs from that of [8] by a factor of -2π.

and,

$$\langle T(z)\,T(w)\rangle = \frac{1}{4\pi^2}\frac{c/2}{(z-w)^4}, \qquad \langle \bar{T}(\bar{z})\,\bar{T}(\bar{w})\rangle = \frac{1}{4\pi^2}\frac{\bar{c}/2}{(\bar{z}-\bar{w})^4}, \tag{2.4}$$

where $c = c_L$ and $\bar{c} = c_R$. Furthermore,

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{T^{\mu}_{\mu}}{4} = 0. \tag{2.5}$$

These results can be found in [7, 8].

3 Coupling the CFT to gravity

Assume that,

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

where,

$$\delta_{z\bar{z}} = \delta_{\bar{z}z} = \frac{1}{2}, \quad \delta_{zz} = \delta_{\bar{z}\bar{z}} = 0, \tag{3.2}$$

and $h_{\mu\nu}$ is a small fluctuation. Using Eq.(2.1), one obtains,

$$\begin{aligned} \delta\,\langle T_{\mu\nu}(z,\bar{z})\rangle &= \delta\int \mathcal{D}[\phi]\,T_{\mu\nu}\,e^{-S} \\ &= \frac{1}{2}\int d^2w\sqrt{\delta}\,\langle T_{\mu\nu}(z,\bar{z})\,T_{\alpha\beta}(w,\bar{w})\rangle\,h^{\alpha\beta}(w,\bar{w}) + \mathcal{O}(h^2). \end{aligned} \tag{3.3}$$

Here, following [8], $d^2z = 2d\sigma^1 d\sigma^2$, where $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$. Using Eqs.(2.2), (2.4) and,

$$\bar{\partial}\frac{1}{z} = \partial\frac{1}{\bar{z}} = 2\pi\delta^2(z), \tag{3.4}$$

it can be verified that,

$$\bar{\partial}\,T = -\frac{1}{4\pi^2}\frac{\pi\,c}{24}\,\partial^3 h^{zz}, \qquad \partial\,\bar{T} = -\frac{1}{4\pi^2}\frac{\pi\,\bar{c}}{24}\,\bar{\partial}^3 h^{\bar{z}\bar{z}}. \tag{3.5}$$

where, e.g. $T = \langle T_{zz}\rangle$ on the curved background (3.1). Eq.(2.5) implies that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, i.e. the non-zero components of the energy momentum tensor on the curved background (3.1), are T and $\bar{T}$. Consequently, Eq.(3.5) means that,

$$\nabla_i T^{ij} \neq 0, \tag{3.6}$$

where, ∇_i denotes the covariant derivative with respect to the Levi-Civita connection corresponding to the metric (3.1), and we have noted that to lowest order in the gravitational field, the covariant derivative can be replaced with ordinary ones.

Eq.(3.6) implies that the energy-momentum tensor is not covariantly conserved. This phenomena is known as gravitational anomaly. To understand why it is called an anomaly, note that an anomaly can be understood as an ultraviolet effect, so it should be a local function. On the other hand the anomaly can not be removed by adding a local term [6]. In our case, Eq.(3.5) implies that

$$T = -\frac{1}{4\pi^2}\frac{\pi\, c}{24}\frac{\partial^3}{\bar{\partial}}h^{zz}, \qquad \bar{T} = -\frac{1}{4\pi^2}\frac{\pi\, \bar{c}}{24}\frac{\bar{\partial}^3}{\partial}h^{\bar{z}\bar{z}}. \tag{3.7}$$

Thus, (1) $\nabla_i T^{ij}$ is a local term, and (2) it can not be eliminated by adding a local term to Eq.(3.7).

Eq.(3.7) can be given in a more suggestive way. Recall that the scalar curvature of the background (3.1) is given by,

$$R = \partial^2 h^{zz} + \bar{\partial}^2 h^{\bar{z}\bar{z}} - 2\partial\bar{\partial}\, h^{z\bar{z}} + \mathcal{O}(h^2), \tag{3.8}$$

and consequently,

$$\frac{\partial}{\bar{\partial}}R = \frac{\partial^3}{\bar{\partial}}h^{zz} + \partial\bar{\partial}h^{\bar{z}\bar{z}} - 2\partial^2\, h^{z\bar{z}} + \mathcal{O}(h^2). \tag{3.9}$$

Thus,

$$T \;=\; -\frac{1}{4\pi^2}\frac{\pi\, c}{24}\frac{\partial}{\bar{\partial}}R + \partial X\ , \tag{3.10}$$

$$\bar{T} \;=\; -\frac{1}{4\pi^2}\frac{\pi\, \bar{c}}{24}\frac{\bar{\partial}}{\partial}R + \bar{\partial} X\ . \tag{3.11}$$

in which,

$$X \;=\; \frac{1}{4\pi^2}\frac{\pi\, c}{24}\left(\bar{\partial}h^{\bar{z}\bar{z}} - 2\partial h^{z\bar{z}}\right), \tag{3.12}$$

$$\bar{X} \;=\; \frac{1}{4\pi^2}\frac{\pi\, \bar{c}}{24}\left(\partial h^{zz} - 2\bar{\partial}h^{z\bar{z}}\right), \tag{3.13}$$

are local terms. These terms can be canceled by means of local counter-terms. To see this, note that by an infinitesimal diffeomorphism,

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{3.14}$$

$R \to R$ and,

$$X \to X' = X - \frac{c}{12\pi}\partial^2\xi_{\bar{z}}, \quad \bar{X} \to \bar{X}' = \bar{X} - \frac{\bar{c}}{12\pi}\bar{\partial}^2\xi_z. \tag{3.15}$$

One can solve $X' = \bar{X}' = 0$ for ξ_z and $\xi_{\bar{z}}$. For example,

$$\xi_{\bar{z}} = \frac{6}{c}\int \frac{z-w}{\bar{z}-\bar{w}}X(w,\bar{w})d^2w. \tag{3.16}$$

Such diffeomorphism renders,

$$R = 4\bar{\partial}^2 h'_{zz} = \partial^2 h'_{\bar{z}\bar{z}}. \tag{3.17}$$

Consequently,

$$\bar{\partial}T = -\frac{1}{4\pi^2}\frac{\pi\, c}{24}\,\partial R, \qquad \partial\bar{T} = -\frac{1}{4\pi^2}\frac{\pi\, \bar{c}}{24}\,\bar{\partial}R, \tag{3.18}$$

$$T_{z\bar{z}} = 0, \qquad T_{\bar{z}z} = 0. \tag{3.19}$$

Exercise: Verify that the corresponding counter-terms are given by $\delta_\xi\, S$.
Hint: Use Eqs.(3.3) and (3.14).

4 Weyl anomlay

Consider adding the following non-covariant counter-term [9],

$$S^{\rm ct} = \beta \int \sqrt{g} R \ln \sqrt{g}, \tag{4.1}$$

which gives,

$$T^{\rm ct}_{\mu\nu} = \beta\, R\, g_{\mu\nu}. \tag{4.2}$$

Thus, for the background (3.1),

$$T^{\rm ct}_{zz} = T^{\rm ct}_{\bar z\bar z} = 0, \quad T^{\rm ct}_{z\bar z} = T^{\rm ct}_{\bar z z} = \frac{\beta}{2} R. \tag{4.3}$$

By choosing $\beta = \frac{1}{4\pi^2}\frac{\pi}{24}(c+\bar c)$ one obtains the following standard result [3],

$$\tilde T_{z\bar z} = \tilde T_{\bar z z} = \frac{\tilde T^\mu_\mu}{4} = \frac{1}{4\pi^2}\frac{\pi}{48}\,(c+\bar c)\,R, \tag{4.4}$$

$$\partial_\mu\,\tilde T^\mu_{\ \nu} = -\frac{1}{4\pi^2}\frac{\pi}{24}(c-\bar c)\,\epsilon^\mu_{\ \nu}\,\partial_\mu\,R. \tag{4.5}$$

where, $\tilde T_{\mu\nu} = T_{\mu\nu} + T^{\rm ct}_{\mu\nu}$. Eq.(4.4) gives the Weyl anomaly. This result is important: if $c = \bar c$, then the gravitational anomaly (3.7) can be eliminated at the cost of Weyl invariance [6].

5 Diffeomorphism Invariance

Assume that, after adding a suitable local counter-term to the action one obtains Eq.(3.18) and,

$$\partial_i\,T^{ij} = 0. \tag{5.1}$$

This implies that,

$$\bar\partial\,T + \partial\,T_{\bar z\,z} = 0, \quad \partial\,\bar T + \bar\partial\,T_{z\bar z} = 0, \tag{5.2}$$

which, gives,

$$T_{\bar z\,z} = \frac{1}{4\pi^2}\frac{\pi\,c}{24}\,R, \quad T_{z\bar z} = \frac{1}{4\pi^2}\frac{\pi\,\bar c}{24}\,R\,. \tag{5.3}$$

Consequently,

$$T_{\bar z\,z} - T_{z\bar z} = \frac{1}{4\pi^2}\frac{\pi}{24}\,(c-\bar c)\,R. \tag{5.4}$$

Thus, if $c \neq \bar c$, diffeomorphism invariance results in anomalous Lorentz symmetry. The exchange of gravitational anomaly for local Lorentz anomaly is discussed in [2].

6 Observing the anomaly

One can show that the gravitational anomaly can be determined in term of the Hawking effect of a 2d Schwarzschild black hole,

$$T_{tt} = \frac{\pi}{12}\,(c+\bar{c})\,T_H^2, \qquad T_{xt} = \frac{\pi}{12}\,(c-\bar{c})\,T_H^2. \tag{6.1}$$

See [3] for details.

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